International Journal of Theoretical Physics, Vol. 7, No. 1 (1973), pp. 9–24

Quantum Field Theory of Fermions Constructed from Bosons. Generalization to Para-Fermi Statistics

A. J. KÁLNAY

Departamento de Física Atómica y Molecular, IVIC, Apartado 1827, Caracas, Venezuela†*; and Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela*

E. MAC COTRINA‡

Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela

and K. V. KADEMOVA

Institute of Physics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Received: 24 *December* 1971

Abstract

It is proved that Fermi fields can, in principle, be constructed in terms of Bose fields, without losing any of the physical properties of the Fermi fields such as, for example, Pauli's Principle. As an example, a Fermi field satisfying Dirac's equation is constructed in terms of Bose fields. Para-Fermi fields are constructed as well. Consequences: Physical theories can, in principle, be reformulated in such a way that Fermions and para-Fermions are described in terms of Bosons.

1. *Introduction*

The basic aim of the present investigation is to show, in principle, that all physical theories that use Fermi quantum fields can be reformulated using only Bose fields. Both theories give the same physical results. For example, as is shown here, in a reformulated theory (i.e. one constructed from Bosons only) Pauli's principle is satisfied by the particles which in the usual theory were introduced as Fermions. These results are extended to parastatistics.§

† Present address.

‡ Holder of a fellowship of the Centro Latinoamericano de Física.

§ In the present paper *parastatistics* refers to systems described in terms of parafields (Greenberg & Messiah, 1965) in contraposition to the analysis of para-symmetries of the wave function (Messiah & Greenberg, 1964).

In Kademova (1970) the above results were obtained in an idealised world with only one Fermi (or para-Fermi) particle. The results were extended in Kademova & Kálnay (1970) for the—again idealised—case of a finite number of Fermions (or para-Fermions). In our present work we shall generalise them to Fermi (or para-Fermi) fields.

In Section 2 we summarise our results concerning Fermi fields only. Parastatistics is briefly reviewed in Section 3. In Sections 4 and 5 we state and prove the theorems (including the extension to parastatistics). As an example a Fermi field satisfying *Dirac equation* is constructed in the Appendix. In order to explain more easily the physical ideas we do not pretend mathematical rigor in the present paper. The transformation properties of the Fermi fields constructed by means of Bose ones and their observables are discussed and found not to be in contradiction with the standard theories (Kálnay & Kademova, to be submitted for publication).

2. *Quantum Theory of Fermions Constructed from Bosons—Summary of Results*

2.1. *Definitions*

Let us choose two fixed natural numbers R and T and two arbitrary sets of complex valued functions $u_{\rho\gamma\zeta}(\mathbf{x})$ and $v_{\tau\Gamma\xi}(\mathbf{z})$ such that each set is orthonormal and complete:

$$\sum_{\zeta=1}^{R} \int d^3x\, u^*_{\rho\gamma\zeta}(\mathbf{x})\, u_{\rho'\gamma'\zeta}(\mathbf{x}) = \delta_{\rho\rho'}\,\delta_{\gamma\gamma'} \tag{2.1.1a}$$

$$\sum_{\gamma=1}^{\infty}\sum_{\rho=1}^{R} u^*_{\rho\gamma\zeta}(\mathbf{x})\, u_{\rho\gamma\zeta'}(\mathbf{x}') = \delta_{\zeta\zeta'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{2.1.1b}$$

$$\rho,\zeta = 1,2,\ldots,R;\ \gamma = 1,2,\ldots$$

$$\sum_{\xi=1}^{T} \int d^3z\, v^*_{\tau\Gamma\xi}(\mathbf{z})\, v_{\tau'\Gamma'\xi}(\mathbf{z}) = \delta_{\tau\tau'}\,\delta_{\Gamma\Gamma'} \tag{2.1.2a}$$

$$\sum_{\Gamma=1}^{\infty}\sum_{\tau=1}^{T} v^*_{\tau\Gamma\xi}(\mathbf{z})\, v_{\tau\Gamma\xi'}(\mathbf{z}') = \delta_{\xi\xi'}\,\delta(\mathbf{z}-\mathbf{z}') \tag{2.1.2b}$$

$$\tau,\xi = 1,2,\ldots,T;\ \Gamma = 1,2,\ldots$$

To each natural number n we put into one-to-one correspondence a pair of natural numbers (ρ_n,γ_n) in the following way:

$$n = R(\gamma_n - 1) + \rho_n, \qquad n = 1,2,3,\ldots, \qquad \gamma_n = 1,2,3,\ldots, \qquad \rho_n = 1,2,\ldots,R \tag{2.1.3}$$

This means that in the arithmetic of natural numbers $\rho_n - 1$ is the remainder and $\gamma_n - 1$ is the quotient of the division of $n-1$ by R. This shows that when n is fixed then the numbers ρ_n and γ_n are unique. Similarly to each natural number i a pair of natural numbers (τ_i, Γ_i) is put into correspondence as follows:

$$i = T(\Gamma_i - 1) + \tau_i, \quad i = 1, 2, 3, \ldots, \qquad \Gamma_i = 1, 2, 3, \ldots, \qquad \tau_i = 1, 2, \ldots, T \tag{2.1.4b}$$

Given a natural number r, we put it into one-to-one correspondence with a sequence,

$$r \leftrightarrow (\mu_1{}^r, \mu_2{}^r, \mu_3{}^r, \ldots), \qquad \mu_k{}^r = 0, 1, \quad r, k = 1, 2, 3, \ldots \tag{2.1.5a}$$

in such a way that

$$r = \sum_{k=1}^{\infty} \mu_k{}^r 2^{k-1} + 1 \tag{2.1.5b}$$

This is the same as to impose that in binary arithmetic the number $r - 1$ be written as

$$\mu^r_{k_{\max}} \mu^r_{k_{\max}-1} \ldots \mu_3{}^r \mu_2{}^r \mu_1{}^r$$

We define a matrix $\overset{+}{F}_\xi{}^1(\mathbf{z})$ with the following complex elements

$$\overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}') = \sum_{i,r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1 - \mu_i{}^r)\, v^*_{\tau_i \Gamma_i \xi}(\mathbf{z}) \times u_{\rho_{r+2^{i-1}}\, \gamma_{r+2^{i-1}}\, \zeta}(\mathbf{x})\, u^*_{\rho_r \gamma_r \zeta'}(\mathbf{x}') \tag{2.1.6}$$

Further we denote by $\overset{+}{a}$ the Hermitian conjugate of a and by c^* the complex conjugate of the number $c \in C$. The sum convention is nowhere used.

2.2. *The Results for Fermi–Dirac Statistics*

Let us start from a Bose field†

$$b_\zeta{}^1(\mathbf{x}), \overset{+}{b}_\zeta{}^1(\mathbf{x}), \qquad \zeta = 1, 2, \ldots, R, \qquad \mathbf{x} \in R^3 \tag{2.2.1}$$

By† $\mathscr{B}^1$ we denote the Fock space of this Bose field, $\mathscr{B}_n{}^1$ denotes the n-particle subspace. The single particle space $\mathscr{B}_1{}^1$ is spanned by the vectors‡

$$|\mathbf{x}, \zeta\rangle^{\mathscr{B}^1} = \overset{+}{b}_\zeta{}^1(\mathbf{x})|0\rangle^{\mathscr{B}^1} \tag{2.2.2}$$

$|0\rangle^{\mathscr{B}^1}$ is the vacuum of the Bose system:

$$b_\zeta{}^1(\mathbf{x})|0\rangle^{\mathscr{B}^1} = 0, \qquad \forall\, \mathbf{x}, \zeta \tag{2.2.3}$$

† The upper index 1 is introduced here for self-consistency with the notation of the following Sections. It means para-statistics of order one, i.e. ordinary statistics.

‡ $|\rangle^{\mathscr{B}^1}_n$ will denote an n-particle Bose as belonging to $\mathscr{B}_n{}^1$. When confusion does not arise the lower index n will be omitted.

We want to construct Fermi fields†

$$f_\xi{}^1(\mathbf{z}), \overset{+}{f}_\xi{}^1(\mathbf{z}), \qquad \xi = 1, 2, \ldots, T, \qquad \mathbf{z} \in R^3$$

as well as their Fock space $\mathscr{F}^1$.

Let us define

$$\overset{+}{f}_\xi{}^1(\mathbf{z}) = \int d^3x \int d^3x' \sum_{\zeta,\zeta'=1}^{R} \overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')\, \overset{+}{b}_\zeta{}^1(\mathbf{x})\, b^1_{\zeta'}(\mathbf{x}') \tag{2.2.4}$$

and‡

$$|0\rangle^{\mathscr{F}^1} = \int d^3x \sum_{\zeta=1}^{R} u_{11\zeta}(\mathbf{x})\, \overset{+}{b}_\zeta{}^1(\mathbf{x})|0\rangle^{\mathscr{B}^1} \tag{2.2.5}$$

It is proved in Section 5 that in the single particle Bose subspace

$$\mathscr{B}_1{}^1 \equiv \mathscr{F}^1 \tag{2.2.6}$$

the operators $\overset{+}{f}_\xi{}^1(\mathbf{z})$ and $f_\xi{}^1(\mathbf{z})$ are Fermi creation and annihilation operators and $|0\rangle^{\mathscr{F}^1}$ is their vacuum state.

Given a Fermi state $|\psi\rangle^{\mathscr{F}^1}$ a complex valued function $g_\zeta{}^\psi(\mathbf{x})$ exists such that

$$|\psi\rangle^{\mathscr{F}^1} = \int d^3x \sum_{\zeta=1}^{R} g_\zeta{}^\psi(\mathbf{x})\, \overset{+}{b}_\zeta{}^1(\mathbf{x})|0\rangle^{\mathscr{B}^1} \tag{2.2.7}$$

The standard Fermi commutation rules are satisfied on $|\psi\rangle^{\mathscr{F}^1}$

$$[f_\xi{}^1(\mathbf{z}), \overset{+}{f}{}^1_{\xi'}(\mathbf{z}')]_+ |\psi\rangle^{\mathscr{F}^1} = \delta_{\xi\xi'}\, \delta(\mathbf{z} - \mathbf{z}')|\psi\rangle^{\mathscr{F}^1} \tag{2.2.8}$$

$$[f_\xi{}^1(\mathbf{z}), f^1_{\xi'}(\mathbf{z}')]_+ |\psi\rangle^{\mathscr{F}^1} = [\overset{+}{f}_\xi{}^1(\mathbf{z}), \overset{+}{f}{}^1_{\xi'}(\mathbf{z}')]_+ |\psi\rangle^{\mathscr{F}^1} = 0 \tag{2.2.8b}$$
$$\forall\ |\psi\rangle^{\mathscr{F}^1} \in \mathscr{F}^1 = \mathscr{B}_1{}^1$$

Let us consider the n-particle Bose

$$|\mathbf{x}_1, \zeta_1; \mathbf{x}_2, \zeta_2; \ldots; \mathbf{x}_n, \zeta_n\rangle^{\mathscr{B}^1} = (n!)^{-1/2}\, \overset{+}{b}{}^1_{\zeta_1}(x_1)\, \overset{+}{b}{}^1_{\zeta_2}(\mathbf{x}_2) \ldots \overset{+}{b}{}^1_{\zeta_n}(\mathbf{x}_n)|0\rangle^{\mathscr{B}^1} \tag{2.2.9a}$$

and Fermi

$$|\mathbf{z}_1, \xi_1; \mathbf{z}_2, \xi_2; \ldots; \mathbf{z}_n, \xi_n\rangle^{\mathscr{F}^1} = (n!)^{-1/2}\, \overset{+}{f}{}^1_{\xi_1}(\mathbf{z}_1)\, \overset{+}{f}{}^1_{\xi_2}(\mathbf{z}_2) \ldots \overset{+}{f}{}^1_{\xi_n}(\mathbf{z}_n)|0\rangle^{\mathscr{F}^1} \tag{2.2.9b}$$

states. The n-particle Fermi state is explicitly constructed as

$$|\mathbf{z}_1, \xi_1; \mathbf{z}_2, \xi_2; \ldots; \mathbf{z}_n, \xi_n\rangle^{\mathscr{F}^1} = (n!)^{-1/2} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n < \infty} \sum_P (-1)^P$$

$$\times \left[\prod_{h=1}^{n} v^*_{\tau_{i_h} \Gamma_{i_h} \xi_{i_h}}(\mathbf{z}_h)\right] \int d^3x \sum_{\zeta=1}^{R} u_{\rho_{1+\sum_{h=1}^{n} 2^{i_h - 1}}\, \gamma_{1+\sum_{h=1}^{n} 2^{i_h - 1}}\, \zeta}(\mathbf{x})\, \overset{+}{b}_\zeta{}^1(\mathbf{x})|0\rangle^{\mathscr{B}^1} \tag{2.2.10}$$

† The upper index 1 is introduced here for self-consistency with the notation of the following Sections. It means para-statistics of order one, i.e. ordinary statistics.

‡ By $|\rangle^{\mathscr{F}^1}$ we denote the Fermi states.

where $(-1)^P$ is the parity of the permutation†

$$P = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

Equation (2.2.10) is *manifestly antisymmetric under interchange of Fermi particles and, as a consequence, Pauli's principle is satisfied.* This is even more transparent in the two Fermi particle states,

$$|\mathbf{z}_1, \xi_1; \mathbf{z}_2, \xi_2\rangle^{\mathscr{F}^1} = 2^{-1/2} \sum_{1 \leqslant i < i' < \infty} [v^*_{\tau_i \Gamma_i \xi}(\mathbf{z})\, v^*_{\tau_{i'} \Gamma_{i'} \xi'}(\mathbf{z}') - v^*_{\tau_i \Gamma_i \xi'}(\mathbf{z}')\, v^*_{\tau_{i'} \Gamma_{i'} \xi}(\mathbf{z})]$$
$$\times \int d^3 x \sum_{\zeta=1}^{R} u_{\rho_{1+2^{i-1}+2^{i'-1}} \gamma_{1+2^{i-1}+2^{i'-1}} \zeta}(\mathbf{x})\, \overset{+}{b}_\zeta{}^1(\mathbf{x}) |0\rangle^{\mathscr{B}^1} \tag{2.2.11}$$

The one Fermi particle states takes the form

$$|z, \xi\rangle^{\mathscr{F}^1} = \sum_{i=1}^{\infty} v^*_{\tau_i \Gamma_i \xi}(\mathbf{z}) \int d^3 x \sum_{\zeta=1}^{R} u_{\rho_{1+2^{i-1}} \gamma_{1+2^{i-1}} \zeta}(\mathbf{x})\, \overset{+}{b}_\zeta(\mathbf{x}) |0\rangle^{\mathscr{B}^1} \tag{2.2.12}$$

The proofs of all the results stated here are particular cases of the theorems of Section 5 which refer to the more general case of parastatistics.

3. *A Short Review of Parastatistics*‡

Consider a numerable set of operators $\overset{+}{b}_r$, b_r satisfying the following commutation relations:

$$[\tfrac{1}{2}[\overset{+}{b}_r, b_s]_+, \overset{+}{b}_t]_- = \delta_{st} \overset{+}{b}_r \tag{3.1a}$$

$$[\tfrac{1}{2}[\overset{+}{b}_r, \overset{+}{b}_s]_+, \overset{+}{b}_t]_- = 0, \qquad \forall\, r, s, t \tag{3.1b}$$

Further on we will be referring to them as to the para-Bose algebra generators.

All the irreducible representations of the commutation relations (3.1) in Fock spaces with unique vacuum state are singled out by the conditions

$$b_r{}^q |0\rangle^{\mathscr{B}^q} = 0, \qquad \forall\, r \tag{3.2}$$

$$b_r{}^q \overset{+}{b}_s{}^q |0\rangle^{\mathscr{B}^q} = q\, \delta_{rs} |0\rangle^{\mathscr{B}^q}, \qquad \forall\, r, s \tag{3.3}$$

where q is a positive integer (order of parastatistics) labelling the irreducible representations of (3.1). The index q to the para-Bose algebra generators fixes the representation. The generators b_r, $\overset{+}{b}_r$ within a fixed representation have an upper index q.

† We stress that n-particle Fermi states are realised as *single* particle Bose states. Cf. equation (2.2.6).

‡ See Green, 1953; Greenberg & Messiah, 1965.

Similarly, by $\overset{+}{f}_i, f_i$ we denote the numerable set of creation and annihilation operators generating a para-Fermi algebra, defined through the commutation relations

$$[\tfrac{1}{2}[\overset{+}{f}_i, f_j]_-, \overset{+}{f}_k]_- = \delta_{jk}\overset{+}{f}_i \tag{3.4a}$$

$$[\tfrac{1}{2}[\overset{+}{f}_i, \overset{+}{f}_j]_-, \overset{+}{f}_k]_- = 0, \qquad \forall\, i,j,k \tag{3.4b}$$

The irreducible representations of the para-Fermi algebra in Fock spaces with unique vacuum state are singled out by the conditions

$$f_i{}^p|0\rangle^{\mathscr{F}^p} = 0, \qquad \forall\, i \tag{3.5}$$

$$f_i{}^p \overset{+}{f}_j{}^p|0\rangle^{\mathscr{F}^p} = p\,\delta_{ij}|0\rangle^{\mathscr{F}^p}, \qquad \forall\, i,j \tag{3.6}$$

p being a positive integer (order of parastatistics) labelling the irreducible representations. As before, the generators $f_i, \overset{+}{f}_i$ within a fixed representation have an upper index p.

Fermi (Bose) algebra is a particular case from the para-Fermi (resp. para-Bose) algebra with order of parastatistics 1.

The Fock space $\mathscr{B}^q$ of a para-Bose algebra of order q is spanned by n-particles kets ($n = 0, 1, 2, \ldots$)

$$|r_1, r_2, \ldots, r_n\rangle^{\mathscr{B}^q} = C^{\mathscr{B}^q}\overset{+}{b}{}^q_{r_1}\overset{+}{b}{}^q_{r_2}\cdots\overset{+}{b}{}^q_{r_n}|0\rangle^{\mathscr{B}^q} \tag{3.7a}$$

similarly the Fock space $\mathscr{F}^p$ of a para-Fermi algebra of order p is spanned by†

$$|i_1, i_2, \ldots, i_n\rangle^{\mathscr{F}^p} = C^{\mathscr{F}^p}\overset{+}{f}{}^p_{i_1}\overset{+}{f}{}^p_{i_2}\cdots\overset{+}{f}{}^p_{i_n}|0\rangle^{\mathscr{F}^p} \tag{3.7b}$$

4. *Second Quantisation of Numerable Para-Fermi Systems with Arbitrary Order of Parastatistics*

Theorem 4.1: The infinite set of matrices $\overset{+}{F}_i{}^1$, $F_i{}^1$, $i = 1, 2, 3, \ldots$, defined through equations

$$(\overset{+}{F}_i{}^1)_{rs} = (-1)^{\sum_{k=1}^{i}\mu_k{}^s}(1-\mu_i{}^s)\,\delta_{\mu_i{}^r,\,\mu_i{}^s+1}\prod_{\substack{l=1\\ l\neq i}}^{\infty}\delta_{\mu_l{}^r,\,\mu_l{}^s} \tag{4.1a}$$

$$(F_i{}^1)_{rs} = (-1)^{\sum_{k=1}^{i}\mu_k{}^r}(1-\mu_i{}^r)\,\delta_{\mu_i{}^r+1,\,\mu_i{}^s}\prod_{\substack{l=1\\ l\neq i}}^{\infty}\delta_{\mu_l{}^r,\,\mu_l{}^s} \tag{4.1b}$$

forms an irreducible representation of a Fermi algebra,

$$([F_i{}^1, \overset{+}{F}{}^1_{i'}]_+)_{rs} = \delta_{rs}\,\delta_{ii'} \tag{4.2a}$$

$$([\overset{+}{F}_i{}^1, \overset{+}{F}{}^1_{i'}]_+)_{rs} = ([F_i{}^1, F^1_{i'}]_+)_{rs} = 0 \tag{4.2b}$$

Proof: It follows from straightforward computation of the anticommutators. It is easier to consider separately the cases $i = i'$, $i < i'$ and $i > i'$. □

† $C^{\mathscr{B}^q}$ and $C^{\mathscr{F}^p}$ are normalisation constants. Their further indices are omitted. See Greenberg & Messiah, 1965. A change in the ordering of the creation operators in equations (3.9) lead, in principle, to different states.

Remarks: (1) The representation (4.1) of the Fermi algebra is irreducible. (2) Notice the smooth generalisation from the Proposition stated in Kademova & Kálnay (1970) for the finite case. The only difference [if equation (4.1) is restricted to finite $2^n \otimes 2^n$ matrices, as in the Reference] is a change of a phase factor $(-1)^{\sum_1^n \mu_k{}^r}$. The phase factor used here can also be used for the finite case, but the opposite is not true.

Note added in proof:

If the matrices $(F_i{}^1)_{rs}$ are rewritten as $(F_i{}^1)_{\mu_1{}^r\mu_2{}^r\ldots;\mu_1{}^s\mu_2{}^s\ldots}$ then, for the finite case, they coincide with the Fermi matrices given by Jordan, P. and Wigner, E. (1928), *Zeitschrift für Physik*, **47**, 631, in their Eq. (69) which was not previously known to us.

Theorem 4.2: Let us define

$$\overset{+}{f}_i = \sum_{r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1 - \mu_i{}^r) \tfrac{1}{2}[\overset{+}{b}{}^q_{r+2^{i-1}}, b_r{}^q]_+ \qquad (4.3a)$$

$$f_i = \sum_{r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1 - \mu_i{}^r) \tfrac{1}{2}[\overset{+}{b}_r{}^q, b^q_{r+2^{i-1}}]_+, \qquad i = 1, 2, 3, \ldots \qquad (4.3b)$$

Then the algebra generated by $\overset{+}{f}_i$ and f_i is a realisation of the para-Fermi algebra in terms of Bose operators.

Proof: Using (4.1), equations (4.3) can be rewritten as

$$\overset{+}{f}_i = \sum_{r,s=1}^{\infty} (\overset{+}{F}_i{}^1)_{rs} \tfrac{1}{2}[\overset{+}{b}_r{}^q, b_s{}^q]_+ \qquad (4.4a)$$

$$f_i = \sum_{r,s=1}^{\infty} (F_i{}^1)_{rs} \tfrac{1}{2}[\overset{+}{b}_r{}^q, b_s{}^q]_+ \qquad (4.4b)$$

The proof immediately follows from a straightforward generalisation of the Theorem given in Kademova (1970) by taking into account Theorem 4.1. □

Note: (1) *The formulae of this Section can also be used for the finite case* by restricting $i = 1, 2, \ldots, n$ and $r, s = 1, 2, \ldots, 2^n$ [cf. Kademova & Kálnay (1970)]. (2) Further in this Section we put $q = 1$ in equations (4.3) so that, *in what follows a realisation of a para-Fermi algebra in terms of Bose operators will be considered.* Then, due to the particular commutation relations of the Bose operators, the Jordan product $\frac{1}{2}[\ ,\ ]_+$ can be replaced by the ordinary operator product in equations (4.3) and (4.4):

$$\overset{+}{f}_i = \sum_{r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1 - \mu_i{}^r) \overset{+}{b}{}^1_{r+2^{i-1}} b_r{}^1 \qquad (4.5a)$$

$$f_i = \sum_{r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1 - \mu_i{}^r) \overset{+}{b}_r{}^1 b^1_{r+2^{i-1}}, \qquad i = 1, 2, 3, \ldots \qquad (4.5b)$$

or simpler

$$\overset{+}{f}_i = \sum_{r,s=1}^{\infty} (\overset{+}{F}_i{}^1)_{rs} \overset{+}{b}_r{}^1 b_s{}^1 \tag{4.6a}$$

$$f_i = \sum_{r,s=1}^{\infty} (F_i{}^1)_{rs} \overset{+}{b}_r{}^1 b_s{}^1 \tag{4.6b}$$

According to the purpose to be used, formulae (4.5) or their equivalents (4.6) are more suitable.

Theorem 4.3: Given a Bose algebra ($q = 1$) with a numerable set of generators $\overset{+}{b}_r{}^1$, $b_r{}^1$ ($r = 1,2,3,\ldots$), the subspace $\mathscr{B}_p{}^1$ of the p-particle Bose states is a Fock space of the para-Fermi algebra with a numerable number of generators $\overset{+}{f}_i{}^p, f_i{}^p$ ($i = 1,2,3,\ldots$), where p is the order of parastatistics.

The vector†

$$|0\rangle^{\mathscr{F}^p} = |p,0,0,\ldots\rangle^{\mathscr{B}^1} \tag{4.7}$$

is a unique vacuum state of the para-Fermi operators.

Remark: The irreducible representation of the para-Fermi algebra is fixed by p and therefore we identify

$$\overset{+}{f}_i|\psi\rangle_p^{\mathscr{B}^1} \equiv \overset{+}{f}_i{}^p|\psi\rangle_p^{\mathscr{B}^1}, \qquad f_i|\psi\rangle_p^{\mathscr{B}^1} \equiv f_i{}^p|\rangle_p^{\mathscr{B}^1}, \qquad \forall|\psi\rangle_p^{\mathscr{B}^1} \in \mathscr{B}_p{}^1 \tag{4.8}$$

Proof: The equations (4.5) imply that

$$\overset{+}{f}_i|\alpha_1,\alpha_2,\alpha_3,\ldots\rangle^{\mathscr{B}^1} = \sum_{r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1-\mu_i{}^r)(\alpha_{r+2^{i-1}}+1)^{1/2}\alpha_r^{1/2}$$
$$\times\, |\alpha_1,\ldots,\alpha_{r-1},\alpha_r-1,\alpha_{r+1},\ldots,\alpha_{r+2^{i-1}-1},\alpha_{r+2^{i-1}}+1,\alpha_{r+2^{i-1}+1},\ldots\rangle^{\mathscr{B}^1} \tag{4.9a}$$

$$f_i|\alpha_1,\alpha_2,\alpha_3,\ldots\rangle^{\mathscr{B}^1} = \sum_{r=1}^{\infty} (-1)^{\sum_{k=1}^{i} \mu_k{}^r} (1-\mu_i{}^r)(\alpha_{r+2^{i-1}})^{1/2}(\alpha_r+1)^{1/2}$$
$$\times\, |\alpha_1,\ldots,\alpha_{r-1},\alpha_r+1,\alpha_{r+1},\ldots,\alpha_{r+2^{i-1}-1},\alpha_{r+2^{i-1}}-1,\alpha_{r+2^{i-1}+1},\ldots\rangle^{\mathscr{B}^1}, \qquad \forall\, i \tag{4.9b}$$

so that

$$f_i|p,0,0,0,\ldots\rangle^{\mathscr{B}^1} = 0, \qquad \forall\, i \tag{4.10}$$

and

$$f_i\overset{+}{f}_j|p,0,0,0,\ldots\rangle^{\mathscr{B}^1} = \delta_{ij}p|p,0,0,0,\ldots\rangle^{\mathscr{B}^1}, \qquad \forall\, i,j \tag{4.11}$$

This completes the proof since equations (4.10) and (4.11) are formally the same as equations (3.5) and (3.6).‡ □

† See notation at the end of Section 3.

‡ The uniqueness of the vacuum follows from equations (3.5) and (4.5) and from Remark 1 to Theorem 4.1.

An abbreviated notation of the basis of $\mathscr{B}_1{}^1$ is

$$|l\rangle_1^{\mathscr{B}^1} = |\delta_{1l}, \delta_{2l}, \delta_{3l}, \ldots\rangle^{\mathscr{B}^1}, \qquad l = 1,2,3,\ldots \tag{4.12}$$

Lemma: The following relations take place

$$\overset{+}{f}_i|l\rangle_1^{\mathscr{B}^1} = (-1)^{\sum_{k=1}^{i} \mu_k{}^l}(1-\mu_i{}^l)|l+2^{i-1}\rangle_1^{\mathscr{B}^1} \tag{4.13a}$$

$$f_i|l\rangle_1^{\mathscr{B}^1} = (-1)^{\sum_{k=1}^{i} \mu_k^{l-2^{i-1}}} \theta(l-2^{i-1})(1-\mu_i^{l-2^{i-1}})|l-2^{i-1}\rangle_1^{\mathscr{B}^1}$$

$$l = 1,2,3,\ldots \tag{4.13b}$$

Proof: The proof follows immediately from equations (4.9). For instance, the proof of the second equation is based on the fact that equation (4.9b) leads to $f_i|l\rangle_1^{\mathscr{B}^1} = 0$ if $l < 2^{i-1}$. This explains the factor $\theta(l-2^{i-1})$, where $\theta(z) = 0$ if $z \leqslant 0$, $\theta(z) = 1$ if $z > 0$. □

Theorem 4.4: The subspace $\mathscr{B}_1{}^1$ of the one-particle Bose states is a Fock space of the Fermi algebra generated by the creation and annihilation operators $\overset{+}{f}_i, f_i$ defined by equations (4.5) [or (4.6)].

Note: As *this theorem is a crucial one for the physically most interesting case* (*description of pure Fermi systems in terms of Bose entities*) we shall give three independent proofs of it.

We must prove that the operators (4.5) or, equivalently, (4.6) satisfy

$$[f_i, \overset{+}{f}_j]_+|\psi\rangle_1^{\mathscr{B}^1} = \delta_{ij}|\psi\rangle_1^{\mathscr{B}^1} \tag{4.14a}$$

and

$$[\overset{+}{f}_i, \overset{+}{f}_j]_+|\psi\rangle_1^{\mathscr{B}^1} = 0, \qquad \forall\, i,j, \qquad \forall\, |\psi\rangle_1^{\mathscr{B}^1} \in \mathscr{B}_1{}^1 \tag{4.14b}$$

First proof: It is well known that the para-Fermi algebra of order 1 is the Fermi algebra. Then, putting $p = 1$ in Theorem 4.3 the proof is completed. □

Second proof: Equations (4.6) imply

$$[f_i, \overset{+}{f}_j]_+ = \sum_{r,r',s,s'=1}^{\infty} (F_i)_{rs}(\overset{+}{F}_j)_{r's'}(\overset{+}{b}_r b_s \overset{+}{b}_{r'} b_{s'} + \overset{+}{b}_{r'} b_{s'} \overset{+}{b}_r b_s) \tag{4.15}$$

so that

$$[f_i, \overset{+}{f}_j]_+|\psi\rangle_1^{\mathscr{B}^1} = ([F_i, \overset{+}{F}_j]_+)_{rs}\overset{+}{b}_r b_s|\psi\rangle_1^{\mathscr{B}^1}, \qquad \forall|\psi\rangle_1^{\mathscr{B}^1} \in \mathscr{B}_1{}^1 \tag{4.16}$$

Then equation (4.14a) follows from equations (4.2a). Equation (4.14b) is proved in the same way. □

Third proof: We shall prove equations (4.14) on each ket belonging to the basis $|l\rangle_1^{\mathscr{B}^1}$ using the previous Lemma and the properties of the numbers $\mu_k{}^r$ introduced in equations (2.1.5). From equations (4.13) we deduce,

$$\overset{+}{f}_j \overset{+}{f}_i |l\rangle_1^{\mathscr{B}^1} = (-1)^{\sum_{k=1}^{j} \mu_k^{l+2^{i-1}} + \sum_{k=1}^{i} \mu_k{}^l} (1-\mu_j^{l+2^{i-1}})(1-\mu_i{}^l)|l+2^{i-1}+2^{j-1}\rangle_1^{\mathscr{B}^1} \tag{4.17a}$$

$$f_i \overset{+}{f}_j |l\rangle_1^{\mathscr{B}^1} = (-1)^{\sum_{k=1}^{i} \mu_k^{l+2^{j-1}-2^{i-1}} + \sum_{k=1}^{j} \mu_k{}^l} \theta(l+2^{j-1}-2^{i-1})(1-\mu_j{}^l) \times (1-\mu_i^{l+2^{j-1}-2^{i-1}})|l+2^{j-1}-2^{i-1}\rangle_1^{\mathscr{B}^1} \tag{4.17b}$$

$$\overset{+}{f}_j f_i |l\rangle_1^{\mathscr{B}^1}\big|_{j\neq i} = (-1)^{\sum_{k=1+\min(i,j)}^{\max(i,j)} \mu_k^{l-2^{i-1}}} \theta(l-2^{i-1})(1-\mu_i^{l-2^{i-1}}) \times (1-\mu_j^{l-2^{i-1}})|l+2^{j-1}-2^{i-1}\rangle_1^{\mathscr{B}^1} \tag{4.17c}$$

and

$$\overset{+}{f}_i f_i |l\rangle_1^{\mathscr{B}^1} = \theta(l-2^{i-1})(1-\mu_i^{l-2^{i-1}})|l\rangle_1^{\mathscr{B}^1} \tag{4.17d}$$

Using these results after tedious but straightforward calculations one gets equations (4.14). □

5. *Second Quantisation of Para-Fermi Fields*

Theorem 5.1*:* Let us call $\overset{+}{F}_\xi{}^1(\mathbf{z})$ the matrices whose elements of indices ζ, $\mathbf{x}$; ζ', $\mathbf{x}'$ are the $\overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z},\mathbf{x},\mathbf{x}')$ introduced in equation (2.1.6). Then the matrices $\overset{+}{F}_\xi{}^1(\mathbf{z})$ and their Hermitian conjugates $\overset{+}{F}_\xi{}^1(\mathbf{z})$ generate an irreducible representation of Fermi algebra.

Proof: Using equations (4.1) one can rewrite the matrix elements of $\overset{+}{F}_\xi{}^1(\mathbf{z})$ as follows,

$$\overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z},\mathbf{x},\mathbf{x}') = \sum_{i,r,s=1}^{\infty} v^*_{\tau_i\Gamma_i\xi}(\mathbf{z})\, u_{\rho_r\gamma_r\zeta}(\mathbf{x})\, u^*_{\rho_s\gamma_s\zeta'}(\mathbf{x}')(\overset{+}{F}_i{}^1)_{rs} \tag{5.1}$$

One completes the proof using Theorem 4.1 (as well as its first Remark), equations (2.1.1) and (2.1.2). □

Lemma: Let us assume that $u_{\rho\gamma\zeta}(\mathbf{x})$ and $v_{\tau\Gamma\xi}(\mathbf{z})$ satisfy, respectively, equations (2.1.1) and (2.1.2). Given the fields $b_\zeta(\mathbf{x})$ and $f_\xi(\mathbf{z})$ (or the operators b_n, f_n) one can construct operators b_r, f_i (resp. fields $b_\zeta(\mathbf{x})$, $f_\xi(\mathbf{z})$) such that the equations

$$\overset{+}{b}_\zeta(\mathbf{x}) = \sum_{r=0}^{\infty} u^*_{\rho_r\gamma_r\zeta}(\mathbf{x})\, \overset{+}{b}_n \tag{5.2a}$$

$$\overset{+}{b}_r = \int d^3x \sum_{\zeta=1}^{R} u_{\rho_r \gamma_r \zeta}(\mathbf{x}) \overset{+}{b}_\zeta(\mathbf{x}), \qquad \zeta = 1, 2, \ldots, R, \qquad \mathbf{x} \in R^3,$$

$$r = 1, 2, 3, \ldots \tag{5.2b}$$

$$\overset{+}{f}_\xi(\mathbf{z}) = \sum_{i=0}^{\infty} v^*_{\tau_i \Gamma_i \xi}(\mathbf{z}) \overset{+}{f}_i \tag{5.3a}$$

and

$$\overset{+}{f}_i = \int d^3z \sum_{\xi=1}^{T} v_{\tau_i \Gamma_i \xi}(\mathbf{z}) \overset{+}{f}_\xi(\mathbf{z}), \qquad \xi = 1, 2, \ldots, T, \qquad \mathbf{z} \in R^3, \qquad i = 1, 2, 3, \ldots \tag{5.3b}$$

as well as their Hermitian conjugates are right. Then:

(i) $b_\zeta(\mathbf{x})$ is a para-Bose field *iff* the b_n are para-Bose operators. The vacuum state $|0\rangle^{\mathscr{B}^q}$ is the same for both and $b_\zeta(\mathbf{x})$ and b_n have the same order of parastatistics q.

(ii) $f_\xi(\mathbf{z})$ is a para-Fermi field *iff* the f_n are para-Fermi operators. The vacuum state $|0\rangle^{\mathscr{F}^p}$ is the same for both and $f_\xi(\mathbf{z})$ and f_i have the same order of parastatistics p.

Proof: Use the notions of parastatistics reviewed in Section 3 as well as equations (2.1.1) and (2.1.2). □

Theorem 5.2: Let us define

$$\overset{+}{f}_\xi(\mathbf{z}) = \int d^3x \int d^3x' \sum_{\zeta,\zeta'=1}^{R} \overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}') \tfrac{1}{2}[\overset{+}{b}_\zeta{}^q(\mathbf{x}), b^q_{\zeta'}(\mathbf{x}')]_+ \tag{5.4a}$$

$$f_\xi(\mathbf{z}) = \int d^3x \int d^3x' \sum_{\zeta,\zeta'=1}^{R} F^1_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}') \tfrac{1}{2}[\overset{+}{b}_\zeta{}^q(\mathbf{x}), b^q_{\zeta'}(\mathbf{x}')]_+,$$

$$\xi = 1, 2, \ldots, T \tag{5.4b}$$

where $b_\zeta{}^q(\mathbf{x})$ is a para-Bose field of order q and $\overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z},\mathbf{x},\mathbf{x}')$ is defined through equation (2.1.6). Then the algebra generated by $\overset{+}{f}_\xi(\mathbf{z})$ and $f_\xi(\mathbf{z})$ is a realisation of a para-Fermi algebra in terms of para-Bose fields $\overset{+}{b}_\zeta(\mathbf{x})$, $b_\zeta(\mathbf{x})$, i.e. the $f_\xi(\mathbf{z}), \overset{+}{f}_\xi(\mathbf{z})$ are para-Fermi fields.†

First proof: It follows immediately from the Theorem given in Kademova (1970), taking into account Theorem 5.1. □

Second proof: It follows from the last Lemma and from Theorem 4.2

† Notice that

$$[\overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')]^* = F^1_{\xi\zeta'\zeta}(\mathbf{z}, \mathbf{x}', \mathbf{x}).$$

For simplicity the proof is sketched in four steps:

(1) Let para-Bose fields $b_\zeta(\mathbf{x})$ ($\mathbf{x} \in R^3, \zeta = 1,2,\ldots,R$) be given.
(2) Using them, we define through equation (5.2b) the para-Bose operators b_r ($r = 1,2,3,\ldots$).
(3) A set of para-Fermi operators f_i ($i = 1,2,3,\ldots$) is realised using the para-Bose operators b_r ($r = 1,2,3,\ldots$) (see Section 4).
(4) Finally we define through equation (5.3a) the para-Fermi fields $f_\xi(\mathbf{z})$ ($\mathbf{z} \in R^3$, $\xi = 1,2,\ldots T$). This gives the explicit realisation (5.4) of the para-Fermi fields $f_\xi(\mathbf{z})$ through the para-Bose field $b_\zeta(\mathbf{x})$. □

Theorem 5.3: Given a Bose algebra ($q = 1$) with the generators $\overset{+}{b}_\zeta{}^1(\mathbf{x})$, $b_\zeta{}^1(\mathbf{x})$ (whose number is equal to the cardinal number of the continuum), $\mathbf{x} \in R^3$, $\zeta = 1, 2, \ldots, R$, then in the p-particle Bose subspace $\mathscr{B}_p{}^1$ a Fock representation of order of parastatistics p of the para-Fermi algebra generated by the fields $\overset{+}{f}_\xi(\mathbf{z}), f_\xi(\mathbf{z})$ [defined by equations (5.4)], whose number also equals the cardinal number of the continuum, is realised, i.e.

$$\overset{+}{f}_\xi(\mathbf{z})|\psi\rangle_p^{\mathscr{B}^1} = \overset{+}{f}_\xi{}^p(\mathbf{z})|\psi\rangle_p^{\mathscr{B}^1}, \qquad f_\xi(\mathbf{z})|\psi\rangle_p^{\mathscr{B}^1} = f_\xi{}^p(\mathbf{z})|\psi\rangle_p^{\mathscr{B}^1}, \qquad \forall\ |\psi\rangle_p^{\mathscr{B}^1} \in \mathscr{B}_p{}^1 \tag{5.5}$$

The vacuum state is

$$|0\rangle^{\mathscr{F}^p} = (p!)^{-1/2} \int d^3x_1 \int d^3x_2 \cdots \int d^3x_p \sum_{\zeta_1,\zeta_2,\ldots,\zeta_p=1}^{R} u_{11\zeta_1}(\mathbf{x}_1)\, u_{11\zeta_2}(\mathbf{x}_2) \cdots \times u_{11\zeta_p}(\mathbf{x}_p) \overset{+}{b}{}^1_{\zeta_1}(\mathbf{x}_1)\, \overset{+}{b}{}^1_{\zeta_2}(\mathbf{x}_2) \cdots \overset{+}{b}{}^1_{\zeta_p}(\mathbf{x}_p)|0\rangle^{\mathscr{B}^p} \tag{5.6}$$

Remark: For $q = 1$ formulae (5.4) reduces to

$$\overset{+}{f}_\xi(\mathbf{z}) = \int d^3x \int d^3x' \sum_{\zeta,\zeta'=1}^{R} \overset{+}{F}{}^1_{\xi\zeta\zeta'}(\mathbf{z},\mathbf{x},\mathbf{x}')\, \overset{+}{b}{}_\zeta^{1}(\mathbf{x})\, b^1_{\zeta'}(\mathbf{x}') \tag{5.7a}$$

$$f_\xi(\mathbf{z}) = \int d^3x \int d^3x' \sum_{\zeta,\zeta'=1}^{R} F^1_{\xi\zeta\zeta'}(\mathbf{z},\mathbf{x},\mathbf{x}')\, \overset{+}{b}{}_\zeta^{1}(\mathbf{x})\, b^1_{\zeta'}(\mathbf{x}') \tag{5.7b}$$

Proof: It follows from Theorem 5.2 and from the last Lemma. To obtain equations (5.7) one must prove that

$$\int d^3x \sum_{\zeta=1}^{R} F^1_{\xi\zeta\zeta}(\mathbf{z},\mathbf{x},\mathbf{x}) = 0 \tag{5.8}$$

and to use equations (5.4) and (2.1.1a). □

Corollary 1. Given an arbitrary state $|\psi\rangle^{\mathscr{F}^p}$ of a para-Fermi field of order of parastatistics p, a complex valued function

$$g^\psi_{\zeta_1,\zeta_2,\ldots,\zeta_p}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$$

exists such that this state can be realised as

$$|\psi\rangle^{\mathscr{F}^p} = \int d^3x_1 \int d^3x_2 \cdots \int d^3x_p \sum_{\zeta_1, \zeta_2, \ldots, \zeta_p = 1}^{R} g^{\psi}_{\zeta_1, \zeta_2, \ldots, \zeta_p}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$$

$$\times \overset{+}{b}{}^1_{\zeta_1}(\mathbf{x}_1) \overset{+}{b}{}^1_{\zeta_2}(\mathbf{x}_2) \cdots \overset{+}{b}{}^1_{\zeta_p}(\mathbf{x}_p)|0\rangle^{\mathscr{B}^1} \tag{5.9}$$

Corollary 2. The results of Section 2.2 follow.

Proof: Put $p = 1$ in Theorem 5.3 and in Corollary 1 (Fermi statistics is a particular case of the para-Fermi one with $p = 1$). To obtain equations (2.2.11) and (2.2.12) use equations (5.4) and (5.5). Then equations (2.2.10) can be proved by induction. □

6. *Discussion*

In the present paper we have constructed an arbitrary para-Fermi field in terms of para-Bose fields. In particular the Fermi fields are realised through Bosons. Their Fock space is the single particle Bose subspace.

What concerns the observables, the equation of motion and the transformation properties of Fermi systems, it will be shown elsewhere (Kálnay & Kademova, to be submitted for publication) that by introducing classical fields with convenient properties, the properties of all Fermi entities realised in terms of Bose entities coincide with those of the theory of Fermi systems. For example, the usual connection between spin and statistics can be achieved. In the present paper we only advance in the Appendix an example concerning the Hamiltonian: we show that a Bose Hamiltonian can be easily constructed such that the Fermi field (realised in terms of Bosons) evolves according to the Dirac equation.

We do not proclaim the naive picture that the Fermions are made up of two Bosons; we construct the *complete* Fermi Fock space on the single particle Bose subspace. This means that *if a suitable Bose system exists, its single particle states can be observed as Fermi states.* Further, the more Bose particle states span Fock spaces of higher order of para-Fermi statistics and therefore they could be perhaps observed as para-Fermions.† If para-Fermions cannot presently be observed one can think that the energy needed to produce multi-Bose states (of the Bosons considered) is too high.

As regards the commutation relations between the Fermi field and the Bose field through which it is expressed (if such Bosons exist) one finds that in the single particle Bose subspace (i.e. in the complete Fermi Fock space) the commutators have zero matrix elements as could be desired. However, when considering the physical meaning of such commutators, care should be taken in order not to count twice one and the same entity: the set of all para-Fermi states is the same as the set of all Bose states.

† The Hamiltonian of the Fermi (para-Fermi) system in terms of Bosons is expressed in bilinear combinations of Bose creation and annihilation operators [see equations (5.7)], this conserves the number of Bosons and does not lead to a change of statistics.

Let us point out that expansions of *pairs* of Fermi operators (Beliav & Zelevinsky, 1962; Jansen *et al.*, 1971; Marumori *et al.*, 1964; Marshalek, 1971) and expansions of odd numbers of Fermi operators by means of Bose operators *plus* a Fermi operator (Banville & Simard, 1970; Shou Yung Li *et al.*, 1971; Simard, 1967, 1969; Yamamura, 1965) have been studied and used.† The present approach has no relation with these attempts since our construction allows to express even a single Fermion through only Bosons. Skyrme (1958, 1961a, 1961b) considered specific models of Bose systems and found that some of their states were Fermi-like; however, they were obtained after some limiting operations and the existence of the limit (i.e. the existence of the Fermi-like operators) was left open. Streater & Wilde (1970) started from Skirme's model and constructed *non-Fock* representations of the canonical commutation relations for a model scalar field theory in two-dimensional space-time; charged fields (with continuous values of the charge) were constructed such that there exist sectors in which Fermi, Bose, non-Fermi and non-Bose commutation relations hold. However, there are not any 'two Fermi particle states' $|\alpha_1, \alpha_2\rangle_2^{\mathscr{F}1}$ (with $\alpha_2 \neq \alpha_1$) because such states are completely described in the 'one Fermi particle states' $|\alpha_1 + \alpha_2\rangle_1^{\mathscr{F}1}$. Our approach refers to general systems (not only to specific models) and to Fock representations; Skirme's limiting process is not used and there is no difficulty (as mentioned above) with any multi-Fermi (or para-Fermi) particle states: different Fermi (or para-Fermi) particle kets are clearly distinguished with the exception of those which (as usual) are represented by a zero vector because Pauli's Principle (and its generalisations) are satisfied.

Let us mention too that expansions of Bose operators in terms of Fermi operators were also known, long ago, in connection with the neutrino theory of light (Jordan, 1935, 1936a, b, c, 1937a, b; Pryce, 1938). Recently a neutrino theory of light has been constructed (Green, 1970) taking the neutrino as a para-Fermion of order two and constructing the photon from it. In relation to these programs (opposite to ours, we start from Bosons), see also Penney (1965a) who pointed out some difficulties for the finite case.

These procedures of realising Bosons through Fermions and vice versa have their analogues in the *c*-number theory where one can construct tensors from spinors as well as describe, by means of tensors, entities (as the electron and the neutrino) which conventionally are described by means of spinors (Whittaker, 1936; Ruse, 1936‡; Penney, 1965b). Moreover, Whittaker (1936) proved that the calculus of relativistic spinors is included in the calculus of tensors. On the other hand the realisations of para-Fermi variables in terms of Bose ones can be retrieved, to a certain extent, in the classical limit (Kálnay, 1972).

† A. O. Barut in the recent preprint IC/72/114 *Fermion States of a Boson Field* (International Centre for Theoretical Physics, Trieste, September 1972) uses a similar creation operator c^+ to obtain a spinor field as a linear combination of products of Bosefields *and* c^+. His work is related to the Streater and Wilde (1970) and not to ours.

‡ We are indebted to Prof. A. Carreño for calling our attention to Ruse's paper.

In a previous paper (Kademova & Kálnay, 1970), devoted to the construction of Fermi (and para-Fermi) creation and annihilation operators in terms of Bose ones for the finite case, it was stated that because of the existence of such a construction '... *a possibility is given for reformulating physical theories in equivalent ones without Fermions (and para-Fermions). The old and the reformulated theories would be physically indistinguishable*'. In the present paper, we proved that for the more realistic case of fields the above-mentioned possibility is right as regards (i) the algebra of the operators and (ii) the state vector space. What remains is to see that the same happens with regard to observables, time evolution and transformation properties, which will be shown by Kálnay & Kademova (to be submitted for publication).

Acknowledgements

A. J. K. is indebted to Professor G. Ruggeri for a critical discussion and E. M. C. is grateful to the Centro Latinoamericano de Física for a fellowship. K. V. K. and A. J. K. are indebted to Professors Abdus Salam and P. Budini, to the IAEA and to the UNESCO for the hospitality kindly extended to them at the International Centre for Theoretical Physics, Trieste, where they had a final discussion of the wording of the paper. A. J. K. is also indebted to them for an Associateship at the Centre.

APPENDIX

Dirac's Equation

Let us consider the field $\overset{+}{f}{}_\xi^1(\mathbf{z})$, $f_\xi^1(\mathbf{z})$ ($\mathbf{z} \in R^3$, $\xi = 1,2,3,4 = T$) of the quantum relativistic electron. Its evolution in time must be consistent with Dirac's equation. As is well known, this is achieved in the standard quantum field theory of Fermi systems (see e.g. Gasiorowicz, 1967) by using the Hamiltonian

$$H_{\mathscr{F}}(\overset{+}{f}{}^1, f^1) = \sum_{\xi,\xi'=1}^{4} \int d^3y \overset{+}{f}{}_\xi^1(\mathbf{y})(-i\nabla_y . \boldsymbol{\alpha} + m\beta)_{\xi\xi'} f_{\xi'}^1(\mathbf{y}) \qquad \text{(A.1)}$$

and by imposing

$$df_\xi^1(\mathbf{z})/dt = i[H_{\mathscr{F}}, f_\xi^1(\mathbf{z})]_- \qquad \text{(A.2)}$$

If we realise the Dirac field through equation (5.1) then the Hamiltonian $H_{\mathscr{F}}(\overset{+}{f}{}^1, f^1)$ is expressed in terms of Bose operators

$$H_{\mathscr{B}}(\overset{+}{b}, b) = \int d^3z \int d^3x \int d^3x' \int d^3x'' \int d^3x''' \sum_{\xi,\xi'=1}^{4} \sum_{\zeta,\zeta',\zeta'',\zeta'''=1}^{R} \overset{+}{F}_{\xi\zeta\zeta'}(\mathbf{z},\mathbf{x},\mathbf{x}')$$
$$\times (-i\nabla_z . \boldsymbol{\alpha} + m\beta)_{\xi\xi'} F_{\xi'\zeta'''\zeta''}(\mathbf{z},\mathbf{x}''',\mathbf{x}'')\, \overset{+}{b}{}_\zeta^1(\mathbf{x})\, b_{\zeta'}^1(\mathbf{x}')\, b_{\zeta''}^1(\mathbf{x}'')\, \overset{+}{b}{}_{\zeta'''}^1(\mathbf{x}''') \qquad \text{(A.3)}$$

of a Bose system up to the term $\mathscr{O}(\overset{+}{b}{}^1, b^1)$ which vanishes in the single particle Bose subspace ($\mathscr{O}(\overset{+}{b}{}^1, b^1)|\psi\rangle_1{}^{\mathscr{B}} = 0$, $\forall\ |\psi\rangle_1{}^{\mathscr{B}} \in \mathscr{B}_1{}^1$).

The equation of motion for the field $f_\xi^1(\mathbf{z})$ computed using the Hamiltonian $H_{\mathscr{B}}$

$$df_\xi^1(\mathbf{z})/dt|\psi\rangle_1^{\mathscr{B}1} = i[H_{\mathscr{B}}, f_\xi^1(\mathbf{z})]_-|\psi\rangle_1^{\mathscr{B}1} \tag{A.4}$$

coincides with the Dirac's equation.

A procedure like the one shown here for Dirac's equation can be used for any other equation of motion of a Fermi system. It has the advantage that it is very simple to construct Hamiltonians like $H_{\mathscr{B}}$, but it always conducts to non-usual Bose Hamiltonians. However, in Kálnay & Mac Cotrina (to be submitted for publication) it will be shown that such unusual Hamiltonians are equivalents (as regards the equations of motion of the Fermi systems) to more standard Bose Hamiltonians.

References

Banville, M. and Simard, P. A. (1970). *Nuclear Physics*, A **158**, 51.
Beliaev, S. T. and Zelevinsky, V. G. (1962). *Nuclear Physics*, **39**, 582.
Green, H. S. (1953). *Physical Review*, **90**, 270.
Green, H. S. (1970). *Parastatistics, Leptons and the Neutrino Theory of Light*. Preprint. The University of Adelaide, Department of Mathematical Physics.
Greenberg, O. W. and Messiah, A. M. L. (1965). *Physical Review* **138**, B 1155.
Jansen, D., Dönan, F., Frauendorf, S. and Jolos, R. V. (1971). *Nuclear Physics* A **172**, 145.
Jordan, P. (1935). *Zeitschrift für Physik*, **93**, 464.
Jordan, P. (1963a). *Zeitschrift für Physik*, **98**, 759.
Jordan, P. (1963b). *Zeitschrift für Physik*, **99**, 109.
Jordan, P. (1963c). *Zeitschrift für Physik*, **102**, 243.
Jordan, P. (1937a). *Zeitschrift für Physik*, **105**, 114.
Jordan, P. (1937b). *Zeitschrift für Physik*, **105**, 229.
Kademova, K. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 2, p. 109.
Kademova, K. and Kálnay, A. J. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 2, p. 115.
Kálnay, A. J. (1972). *International Journal of Theoretical Physics*, Vol. 6, No. 6, p. 415.
Kálnay, A. J. and Kademova, K. V. (1973). *Observables and Transformation Properties of Fermions Constructed in Terms of Bosons*. (To be submitted for publication.)
Kálnay, A. J. and MacCotrina, E. (1973). *A Remark on the Observables of Fermi Systems Described in Terms of Bosons*. (To be submitted for publication.)
Marshalek, E. R. (1971). *Nuclear Physics*, A **161**, 401.
Marumori, T., Yamamura, M. and Tokunaga, A. (1964). *Progress of Theoretical Physics*, **31**, 1009.
Messiah, A. M. L. and Greenberg, O. W. (1964). *Physical Review*, **136**, B 248.
Penney, R. (1965a). *Journal of Mathematical Physics*, **6**, 1026.
Penney, R. (1965b). *Journal of Mathematical Physics*, **6**, 1031.
Pryce, M. H. L. (1938). *Proceedings of the Royal Society* (*London*), A **165**, 247.
Ruse, H. S. (1936). *Proceedings of the Royal Society* (*Edinburgh*), **57**, 97.
Shou Yung Li, Dreizler, R. M. and Klein, A. (1971). *Physical Review C*, **4**, 1571.
Simard, P. A. (1967). *Canadian Journal of Physics*, **45**, 3241.
Simard, P. A. (1969). *Canadian Journal of Physics*, **47**, 103.
Skyrme, T. H. R. (1958). *Proceedings of the Royal Society* (*London*), A **247**, 260.
Skyrme, T. H. R. (1961a). *Proceedings of the Royal Society* (*London*), A **260**, 127.
Skyrme, T. H. R. (1961b). *Proceedings of the Royal Society* (*London*), A **262**, 237.
Streater, R. F. and Wilde, I. F. (1970). *Nuclear Physics B*, **24**, 561.
Whittaker, E. T. (1936). *Proceedings of the Royal Society* (*London*), A **158**, 38.
Yamamura, M. (1965). *Progress of Theoretical Physics*, **33**, 199.